Introduction to Small Area Estimation and Poverty Mapping

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Lecture Plan

- Introduction to small area estimation
- Direct and model-assisted estimation
- Introduction to model-based methods
  - Unit-level models - The Battese-Harter-Fuller model
  - Area-level models - The Fay-Herriot model
  - Methods for non-linear statistics
  - Applications using R

1 – Introduction to SAE, Direct & Model-assisted Estimation

Introduction

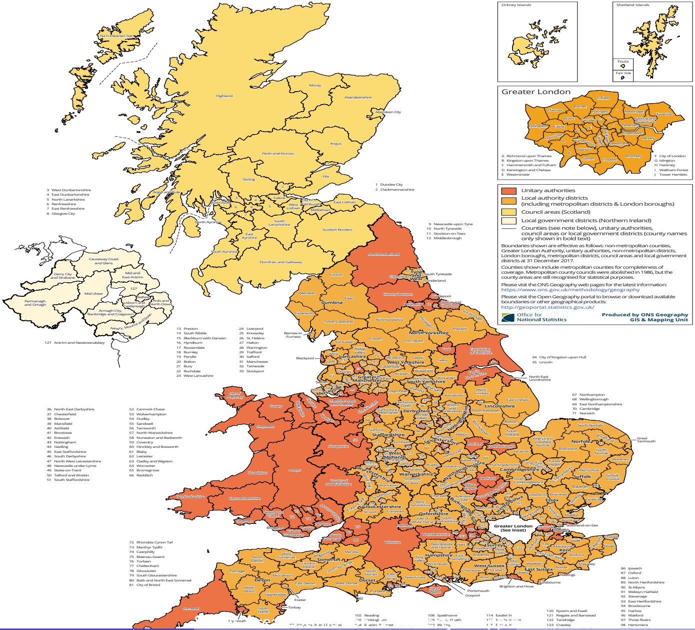
- Statistical models are used to study the relationship between a response variable and a set of predictor variables
- However, data (e.g. survey data) are also used to estimate *finite population parameters*
- Examples: Average income; unemployment rate; at-risk-of-poverty rate for
    - Large populations and
    - Smaller sub-populations (areas/domains)

Introduction

- Estimates for small areas are referred to as *small area statistics*
- The research field is called small area estimation (SAE)

- Areas or Domains often correspond to
  - Socio-demographic groups: E.g. age by gender groups
  - Geographic domains (areas): Provinces, municipalities, school districts, health service areas

UK: Local authority districts, counties and unitary authorities,[1] 2017



Demand for small area statistics

- Demand for area statistics has increased due to their use in
  - Formulating social and economic policies
  - Allocation of government funds
  - Regional planning
  - Business decision making (e.g. many small businesses rely on information about local socio-economic conditions)
- SAE is a fast growing area of methodological research
- Productive cooperation between academia and practitioners
- Significant progress with uptake of SAE methods by National Statistical Institutes

Initial triplet of estimates

1. Direct estimators of the mean

- Hajek-Brewer Ratio estimator (No auxiliary information)

$$\hat{\bar{\theta}}_i^{Direct} = (\sum_{j=1}^{n_i} y_{ij}/\pi_{ij})/(\sum_{j=1}^{n_i} 1/\pi_{ij})$$

- GREG estimator (Auxiliary information)

$$\hat{\bar{\theta}}_{i,GREG}^{Direct} = \frac{1}{N_i}\sum_{j=1}^{n_i} w_{ij}y_{ij}, w_{ij} = g_{ij}/\pi_{ij},$$

$$g_{ij} = 1 + (X - \sum_i \sum_{j=1}^{n_i} \mathbf{x}_{ij}/\pi_{ij})^T(\sum_i \sum_{j=1}^{n_i} \mathbf{x}_{ij}\mathbf{x}_{ij}^T/\pi_{ij})^{-1}\mathbf{x}_{ij}$$

Initial triplet of estimates

2. Synthetic estimators

- No auxiliary information

$$\hat{\theta}_i^{Synthetic} = \hat{\theta}_L$$

- Auxiliary information

$$\hat{\theta}_i^{Synthetic} = \bar{\boldsymbol{x}}_i^T \hat{\boldsymbol{\beta}},$$

$$\hat{\boldsymbol{\beta}} = (\sum_i \sum_{j=1}^{n_i} \mathbf{x}_{ij}\mathbf{x}_{ij}^T/\pi_{ij})^{-1}(\sum_i \sum_{j=1}^{n_i} \mathbf{x}_{ij} y_{ij}/\pi_{ij})$$

3. Composite estimator

$$\hat{\theta}_i^{Composite} = \alpha_i \hat{\theta}_i^{Direct} + (1 - \alpha_i)\hat{\theta}_i^{Synthetic}$$

Complex indicators: Head count ratio

- The Head Count ratio (HCR) also known as the at-risk-of-poverty-rate (ARPR).
- The HCR depends on a poverty threshold (at-risk-of-poverty threshold, ARPT), which is set at 60% of the national median income.

$$\widehat{ARPT} = 0.6 \cdot \hat{q}_{0.5},$$

where $\hat{q}_{0.5}$ is the median.

$$\widehat{HCR} = \frac{\sum_j I(y_j < \widehat{ARPT}) w_j}{\sum_{j=1}^{n} w_j} \cdot 100$$

Complex indicators (Income inequality): Quintile Share Ratio

For a given sample, let $\hat{q}_{0.2}$ and $\hat{q}_{0.8}$ denote the weighted 20% and 80% quantiles, respectively. Using index sets $I_{\leq \hat{q}_{0.2}}$ and $I_{> \hat{q}_{0.8}}$, the quintile share ratio is estimated by

$$\widehat{QSR} = \frac{\sum_{j \in I_{> \hat{q}_{0.8}}} w_j y_j}{\sum_{j \in I_{\leq \hat{q}_{0.2}}} w_j y_j}.$$

EU-SILC survey: Austria

- The *European Union Statistics on Income and Living Conditions* (EU-SILC) is one of the most well-known panel surveys and is conducted in EU member states and other European countries.
- It is used as a basis for computing *Laeken indicators*, a set of indicators for measuring risk-of-poverty in Europe. In particular,
  - Inequality: Quintile share ratio or Gini coefficient.
  - Poverty: At-risk-of-poverty-rate (head count ratio) or Poverty Gap.
- The survey serves as a starting point for the Europe 2020 strategy for smart, sustainable and inclusive growth.

Reference: Alfons et al. (2011); Alfons and Templ (2013)

Datasets: Austrian EU-SILC

- The dataset contains 14,827 observations from 6000 households.
- Sample consists of 28 most important variables containing information on
  - Demographics
  - Income
  - Living conditions
- The data are synthetically generated from the original Austrian EU-SILC data from 2006.

Reference: Alfons et al. (2011); Alfons and Templ (2013)

Austrian EU-SILC variables

Variable	Name
Equivalized household income	eqIncome
Region	db040
Household ID	db030
Household size	hsize
Age	age
Gender	rb090
Self-defined current economic status	pl030
Citizenship	pb220a
Employee cash or near cash income	py010n
Cash benefits or losses from self-employment	py050n
Unemployment benefits	py090n
Old-age benefits	py100n
Equivalized household size	eqSS

Reference: Alfons et al. (2011); Alfons and Templ (2013)

Income dataset from Mexico

- The data covers one of the 32 federal entities in Mexico; State of Mexico (EDOMEX).
- Household level survey data with income outcomes and potential covariates (ENIGH survey).
- Survey uses a stratified simple random cluster sample.
- The law requires access to estimates for each municipality.
- 125 municipalities in EDOMEX, 58 are part of the sample, 67 are out of sample.
- The survey includes 2748 households and 115 variables.

Reference: CONEVAL (2010)

Mexico and the State of Mexico

Computation - Direct domain estimation with *R*

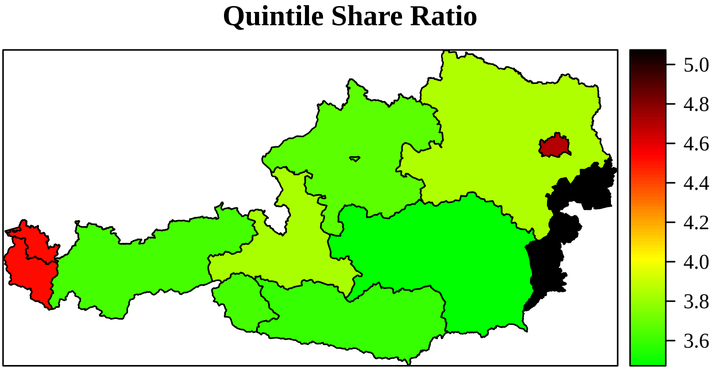
- *R* package `laeken` can be used for direct estimation of linear and non-linear domain indicators
- One feature of `laeken` is that indicators can be computed for different subdomains (regions, age or gender).
- All the user needs to do is to specify such a categorical variable via the `breakdown` argument.
- Note that for the Head count ratio, the same overall at-risk-of-poverty threshold is used for all subdomains.

Using R-package laeken: QSR at domain level

```
> # QSR - breakdown by NUTS2
> qsr("eqIncome", weights = "rb050", data = eusilc,
   breakdown="db040")
Value:
[1] 3.971415

Value by domain:
         stratum    value
1     Burgenland 5.073746
2      Carinthia 3.590037
3  Lower Austria 3.845026
4       Salzburg 3.829411
5         Styria 3.472333
6          Tyrol 3.628731
7  Upper Austria 3.675467
8         Vienna 4.705347
9     Vorarlberg 4.525096
```

Reference: Alfons and Templ (2013)

Quintile share ratio breakdown by NUTS2



National Quintile share ratio: 3.97

Variance estimation

Measures of uncertainty

- Variance,
- Coefficient of Variation

Variance estimation

Resampling methods

- Jackknife
- Bootstrap

Analytic methods

- Taylor linerisation

Using R-package laeken: Variance estimation

```
> hcr_nuts2<- arpr("eqIncome", weights = "rb050",
   breakdown = "db040", data = eusilc)
> variance("eqIncome", weights = "rb050", breakdown = "
   db040", design = "db040",
+           data = eusilc, indicator = hcr_nuts2, bootType
    = "naive", seed = 123,R=500)

Value by domain:
        stratum    value
1    Burgenland 19.53984
2     Carinthia 13.08627
3 Lower Austria 13.84362
...
6         Tyrol 15.30819
7 Upper Austria 10.88977
8        Vienna 17.23468
9    Vorarlberg 16.53731
```

Reference: Alfons and Templ (2013)

Using R-package `laeken`: Variance estimation

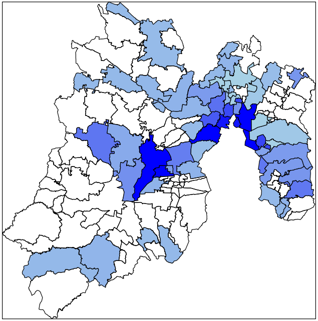
```
Variance by domain:
        stratum       var
1    Burgenland 3.2426875
2     Carinthia 1.2348834
...
7 Upper Austria 0.3499630
8        Vienna 0.5600269
9    Vorarlberg 2.0032567

Confidence interval by domain:
        stratum     lower    upper
1    Burgenland 16.296501 23.13324
2     Carinthia 10.679302 15.24175
...
7 Upper Austria  9.720091 12.07298
8        Vienna 15.662437 18.62901
9    Vorarlberg 13.560864 19.14820
```

Reference: Alfons and Templ (2013)

Problems with direct estimation

- Often the sample not large enough for domain estimation
- Design of the survey does not account for competing interests regarding the targets of estimation
- Not all domains of interest include sampled units
- Small sample sizes $\rightarrow$ Large variance of direct estimates

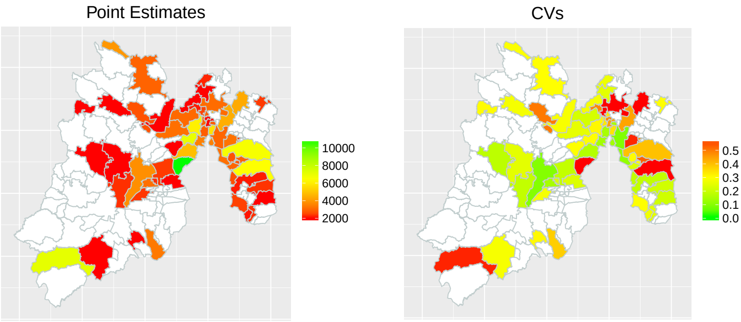
An example: Poverty mapping in Mexico



- 125 municipalities in state of Mexico. Only 58 are included in the survey
- For the municipalities in the sample, the average sample size is 47 households

An example: Poverty mapping in Mexico

Direct estimates of average household equivalised income and CVs



2 – Small Area Estimation - Model-based methods

Recap

- Domain: sub-population of the population of interest planned or unplanned
  - Geographic areas (e.g. Regions, Provinces, Municipalities, Health Service Area)
  - Socio-demographic groups (e.g. Sex, Age, Race within a large geographic area)
  - Other sub-populations (e.g. the set of firms belonging to an industry subdivision)

Direct estimators may be unreliable due to small sample sizes

Unit-level models: Battese-Harter-Fuller model

Key Concept:

Include random area-specific effects to account for between area variation/ unexplained variability between the small areas.

Random effects model:

Notation: (i =domain, j =individual)

$$y_{ij} = \mathbf{x}_{ij}^T \boldsymbol{\beta} + u_i + e_{ij}, j = 1, ..., n_i, i = 1, ..., m$$

- Random effects $u_i \sim N(0, \sigma_u^2)$
- Error term $e_{ij} \sim N(0, \sigma_e^2)$

Reference: Battese et al. (1988)

Unit-level models: Battese-Harter-Fuller model

Empirical Best Linear Unbiased Predictor (EBLUP) of $\bar{y}_i$ is

$$\hat{\theta}_i^{BHF} = \hat{\bar{y}}_i = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} \hat{y}_{ij}\Big\} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} (\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}} + \hat{u}_i)\Big\}$$

where

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}^T\hat{\boldsymbol{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\hat{\boldsymbol{V}}^{-1}\boldsymbol{y} \\
\hat{\boldsymbol{u}} &= \hat{\sigma}_u^2\boldsymbol{Z}^T\hat{\boldsymbol{V}}^{-1}(\boldsymbol{y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \\
\hat{\boldsymbol{V}} &= \hat{\sigma}_u^2\boldsymbol{Z}\boldsymbol{Z}^T + \hat{\sigma}_e^2\boldsymbol{I}_n
\end{aligned}$$

Reference: Battese et al. (1988)

Analytic MSE estimation: The Battese-Harter-Fuller model

An MSE estimator of the small area estimator of the mean under BHF is (see Prasad & Rao, 1990)

$$MSE(\hat{\theta}_i^{BHF}) = g_{1i} + g_{2i} + g_{3i}$$

- g_{1i}, g_{2i} uncertainty of BLUP, treating variance components as known
- g_{3i} uncertainty due to estimation of the variance components

Remark: Alternatively (for more complex models) use bootstrap or jackknife methods

Reference: Prasad and Rao (1990)

Using R-package sae: The Battese-Harter-Fuller model

Using the EU-SILC data

```
> # Direct estimation of mean using sae-package
> fit_direct<-direct(y=eqIncome,dom=region,data=eusilcS_
   HH,replace=T)
>
> # Estimation of the Unit-level model (Battese-Harter-
   Fuller)
> fit_EBLUP<-eblupBHF(formula=as.numeric(eqIncome)~py010n
    + py050n+hy090n,dom=region,data=eusilcS_HH,meanxpop=
   Xmean,popnsize=Popsize)
>
> # MSE estimation of the Unit-level model
> MSE_EBLUP<-pbmseBHF(formula=as.numeric(eqIncome)~py010n
    + py050n+hy090n,dom=region,data=eusilcS_HH,meanxpop=
   Xmean,popnsize=Popsize)
```

Reference: Molina and Marhuenda (2015)

Using R-package sae: The Battese-Harter-Fuller model

```
> # Comparison of direct and EBLUP
        Domains   Direct EBLUP_est    CV EBLUP_CV
     Burgenland 15781.61  20954.84 18.45     5.47
  Lower Austria 20476.21  20727.56  6.45     5.21
         Vienna 18996.19  21022.50  5.09     5.39
      Carinthia 20345.62  20526.51  9.01     5.74
         Styria 21184.01  20839.66  6.64     5.42
  Upper Austria 21074.00  21433.11  5.36     5.75
       Salzburg 18716.99  20841.91  7.41     5.74
          Tyrol 18060.43  20805.72 10.38     5.32
     Vorarlberg 18922.28  22028.77 10.69     5.93
```

Reference: Molina and Marhuenda (2015)

Area-level models: The Fay-Herriot model

Sampling model

$$\hat{\theta}_i^{direct} = \theta_i + e_i$$

- $\hat{\theta}_i^{direct}$ is a direct design-unbiased estimator, for instance the Horvitz-Thompson / Brewer estimator
- e_i is the sampling error of the direct estimator

Linking model

$$\hat{\theta}_i^{direct} = \mathbf{x}_i \boldsymbol{\beta} + u_i + e_i, \quad i = 1, \dots, m,$$

where $u_i \sim N(0, \sigma_u^2)$ and $e_i \sim N(0, \sigma_{e_i}^2)$, with $\sigma_{e_i}^2$ assumed known

Reference: Fay and Herriot (1979)

Area-level models: The Fay-Herriot estimator

The EBLUP under the **Fay-Herriot** (FH) model is obtained by

$$\begin{aligned}\hat{\theta}_i^{FH} &= \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}} + \hat{u}_i \\ &= \gamma_i \hat{\theta}_i^{direct} + (1 - \gamma_i) \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}\end{aligned}$$

Analytic MSE estimation: The Fay-Herriot model

An MSE estimator of the small area estimator of the mean under BHF is (see Prasad & Rao, 1990)

$$MSE(\hat{\theta}_i^{BHF}) = g_{1i} + g_{2i} + g_{3i}$$

- g_{1i}, g_{2i} uncertainty of BLUP, treating variance components as known
- g_{3i} uncertainty due to estimation of the variance components

Remark: Alternatively (for more complex models) use bootstrap or jackknife methods

Reference: Prasad and Rao (1990)

Using R-package sae: Fay-Herriot

Based on a synthetic population

```
> # Direct estimation of mean using sae-package
> fit_direct<-direct(y=eqIncome,dom=region,data=eusilcS_
   HH,replace=T)
>
> # Aggregation of the covariates on region level
> eusilcP_HH_agg<-tbl_df((eusilcP_HH))%>%group_by((region
   ))%>%summarise(hy090n=mean(hy090n))%>%
+ ungroup()%>%mutate(Domain=fit_direct$Domain)
>
> # Merging the datasets
> data_frame<-left_join(eusilcP_HH_agg,fit_direct,by="
   Domain")%>%mutate(var=SD^2)
>
> # Estimation of the FH-model
> fit_FH<-mseFH(formula=Direct ~ hy090n,vardir=var,data=
   as.data.frame(data_frame))
```

Reference: Molina and Marhuenda (2015)

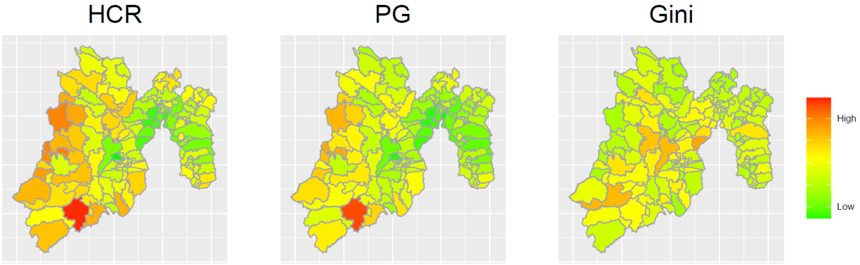
Using R-package sae: Fay-Herriot

```
> # Comparison of direct and FH
         Domains SampSize   Direct   FH_est    CV FH_CV
      Burgenland       14 15781.61 16595.25 18.45 12.29
   Lower Austria       71 20476.21 19912.64  6.45  5.14
          Vienna       95 18996.19 20135.40  5.09  6.65
       Carinthia       34 20345.62 20260.46  9.01  4.30
          Styria       46 21184.01 20541.93  6.64  5.33
   Upper Austria       67 21074.00 19702.94  5.36  5.84
        Salzburg       26 18716.99 18908.88  7.41  5.82
           Tyrol       32 18060.43 19729.34 10.38  4.01
      Vorarlberg       15 18922.28 18342.81 10.69  6.22
```

Reference: Molina and Marhuenda (2015)

3 – Small Area Estimation of non-linear indicators

Typical results of poverty mapping



Non-linear indicators

- Small area estimation methods mainly focus on estimating means and proportions
- New developments in SAE methodologies focus on estimating non-linear statistics e.g poverty/inequality indicators
- Methodology is general and covers linear and non-linear indicators

Data Requirements

- Estimation requires access to unit-level population covariates (e.g. Census microdata)
- Data access is challenging
- Use of area-level models is possible
- Here we focus on unit-level models

Recent methodologies

- The World Bank method (ELL)
  (Elbers et al., 2003)
- The Empirical Best Predictor (EBP) method
  (Molina and Rao, 2010)
- Methods based on M-Quantiles
  (Tzavidis et al., 2010)

Empirical Best Prediction (EBP)

$$y_{ij} = \mathbf{x}_{ij}^T \boldsymbol{\beta} + u_i + e_{ij}, \quad j = 1, \ldots, n_i, \quad i = 1, \ldots, D,$$

1. Use the sample data to estimate $\hat{\beta}, \hat{\sigma}_u^2, \hat{\sigma}_e^2, \hat{u}_i$ and $\hat{\gamma}_i = \frac{\hat{\sigma}_u^2}{\hat{\sigma}_u^2 + \frac{\hat{\sigma}_e^2}{n_i}}$.
2. For $l = 1, \ldots, L$
   - Compute $E(y_r|y_s)$ under the assumption of normal errors
   - Generate $e_{ij}^* \sim N(0, \hat{\sigma}_e^2)$ and $u_i^* \sim N(0, \hat{\sigma}_u^2 \cdot (1 - \hat{\gamma}_i))$, simulate a pseudo-population

     $$y_{ij}^{*(l)} = \mathbf{x}_{ij}^T \hat{\beta} + \hat{u}_i + u_i^* + e_{ij}^*$$

   - Calculate the measures of interest, e.g. poverty indicator, $\theta_i^{(l)}$.
3. Obtain $\hat{\theta}_i^{EBP} = 1/L \sum_{l=1}^{L} \hat{\theta}_i^{(l)}$ for each area i.

Reference: Molina and Rao (2010).

Parametric bootstrap: MSE estimation

- Fit the random effects model to the original sample
- Generate $u_i^* \sim N(0, \hat{\sigma}_u^2)$, $e_{ij}^* \sim N(0, \hat{\sigma}_e^2)$
- Construct B bootstrap populations

$$y_{ij}^* = \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}} + u_i^* + e_{ij}^*$$

- For each b population compute the population value θ_i^{*b}
- From each bootstrap population select a bootstrap sample
- Implement the EBP with the bootstrap sample, get $\hat{\theta}_i^{*b}$

$$\widehat{MSE}(\hat{\theta}_i) = B^{-1} \sum_{b=1}^{B} (\hat{\theta}_i^{*b} - \theta_i^{*b})^2$$

Reference: González-Manteiga et al. (2008).

Using R-package `emdi`: EBP method

- The R-package `emdi` is an alternative to the `sae` package
- `sae` currently includes more methods
- `emdi`, user friendly, more emphasis on presentation of the results
- Estimation for linear and non-linear indicators using unit-level models
- Includes data-driven transformations
- Currently being extended to include area-level models

- The R package **emdi** includes two synthetic data sets
  - `eusilcS_HH`: sample data from Austrian regions about household income and demographics
  - `eusilcP_HH`: population micro-data for the Austrian regions
  - Both data sets contain the same covariates

Reference: Kreutzmann et al. (2019).

Using R-package emdi: EBP method

Implemented in the R package **emdi** via function `ebp()`

```
# EBP estimation function
ebp_au <- ebp(fixed = eqIncome ~ gender + eqsize +
                      py010n + py050n + py090n +
                      py100n + py110n + py120n +
                      py130n + hy040n + hy050n +
                      hy070n + hy090n + hy145n,
              pop_data = eusilcP_HH,
              pop_domains = "region",
              smp_data = eusilcS_HH,
              smp_domains = "region",
              pov_line = 0.6*median(eusilcS_HH$eqIncome
                 ),
              transformation = "no",
              L=50,
              MSE = T,
              B = 50)
```

Reference: Kreutzmann et al. (2019).

Using R-package emdi: EBP method - Summary output

```
# Summary for the EBP method
  > summary(ebp_au)
```

```
  Out-of-sample domains:  0
  In-sample domains:  9

  Sample sizes:
  Units in sample:  503
  Units in population:  25000
                 Min. 1st Qu. Median  Mean  3rd Qu. Max.
  Sample_domains   16   26      43    55.9    94    101
  Pop_domains     799 1671    1889    2778  4071   5857
```

Reference: Kreutzmann et al. (2019).

Using R-package emdi: EBP method - Summary output

```
Explanatory measures:
Marginal_R2 Conditional_R2
0.5198029      0.5198029

Residual diagnostics:
             Skewness Kurtosis Shapiro_W    Shapiro_p
Error         2.17646 12.5925  0.8551573   4.0933e-21
Random_effect 0.64311  2.6048  0.8870226   1.8589e-01

ICC:  2.610126e-08
```

Reference: Kreutzmann et al. (2019).

Motivating alternative methods

- EBP relies on Gaussian assumptions :
  - ✓ $u_i \overset{iid}{\sim} N(0, \sigma_u^2)$, the random area-specific effects
  - ✓ $e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$

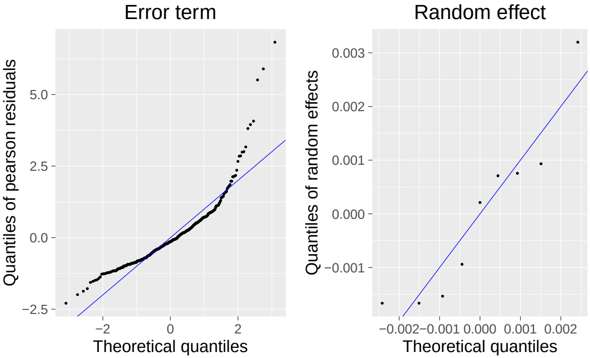
Model Checking (Residual diagnostics)

- Q-Q plots of residuals at different levels
- Influence diagnostics
- Plot standardised residuals vs fitted values - Heteroscedasticity

Graphical investigation of normality

Q-Q plots can help to assess the normality assumptions and it belongs to one of the plots that are automatically provided when applying the function `plot` to an `emdi` object

```
# Residual diagnostics
    > plot(ebp_au)
```



Model adaptations

- Use an EBP formulation under an alternative distribution (Graf et al., 2015) - Model under generalised Beta distribution of the second kind
- Use robust methods as an alternative to transformations (Chambers and Tzavidis, 2006; Ghosh, 2008; Sinha and Rao, 2009; Chambers et al., 2014; Schmid et al., 2016)
- Use non-parametric models (Opsomer et al., 2008; Ugarte et al., 2009)
- Elaborate the random effects structure e.g. include spatial structures (Pratesi and Salvati, 2008; Schmid et al., 2016)
- **Use of transformations** (Rojas-Perilla et al., 2019)

Why transformations might help?

- Attempt to satisfy the model assumptions:
  - Normality: Reducing skewness and controlling kurtosis
  - Homoscedasticity: Variance-stabilization
  - Linearity: linearizing relation between variables

Some choices of transformations

- Shifted transformations
  - Log-shift
- Power transformations
  - Box-Cox
  - Exponential
  - Sign power
  - Modulus
  - Dual power
  - Convex-to-concave
- Multi-parameter transformations
  - Johnson
  - Sinh-arcsinh

Scaled transformations

Scaled Log-Shift Transformation (λ)

$$T_\lambda(y_{ij}) = \alpha \log(y_{ij} + \lambda),$$

Scaled Box-Cox Transformation (λ)

$$T_\lambda(y_{ij}) = \begin{cases} \frac{(y_{ij}+s)^\lambda - 1}{\alpha^{\lambda-1}\lambda}, & \lambda \neq 0 \\ \alpha \log(y_{ij} + s), & \lambda = 0 \end{cases},$$

Scaled Dual Power Transformation (λ)

$$T_\lambda(y_{ij}) = \begin{cases} \frac{2}{\alpha} \frac{(y_{ij}+s)^\lambda - (y_{ij}+s)^{-\lambda}}{2\lambda} & \text{if } \lambda > 0; \\ \alpha \log(y_{ij} + s) & \text{if } \lambda = 0. \end{cases}$$

with α chosen in such that the Jacobian of the transformation is 1

Reference: Rojas-Perilla et al. (2019).

Estimation methods of (λ) for linear mixed models

- Skewness minimization
- Divergence minimization
- ML/REML

Estimation algorithm (λ)

REML Algorithm for the EBP Method:

1. Choose a transformation type
2. Define a parameter interval for λ
3. Set λ to a value inside the interval
4. Maximize the residual log-likelihood function conditional on fixed λ
5. Repeat 3 and 4 until maximum until $\hat{\lambda}$ is found
6. Apply the EBP method using $\hat{\lambda}$

Reference: Rojas-Perilla et al. (2019).

Parametric bootstrap for MSE estimation

1. For $b = 1, ..., B$
   - Using the already estimated $\hat{\beta}, \hat{\sigma}_u^2, \hat{\sigma}_e^2, \hat{\lambda}$ from the transformed data $T(y_{ij}) = \tilde{y}_{ij}$, simulate a bootstrap superpopulation $\tilde{y}_{ij}^{*(b)} = \mathbf{x}_{ij}^T \hat{\beta} + u_i^* + e_{ij}^*$
   - Transform $\tilde{y}_{ij}^{*(b)}$ to original scale resulting in ${y_{ij}}^{*(b)}$
   - For each b population compute the population value θ_i^{*b}
   - Extract the bootstrap sample in ${y_{ij}}^{*(b)}$ and use the EBP method
   - Estimate λ with the bootstrap sample
   - Obtain $\hat{\theta}_i^{*b}$
2. $\widehat{MSE}(\hat{\theta}_i) = B^{-1} \sum_{b=1}^{B} (\hat{\theta}_i^{*b} - \theta_i^{*b})^2$

Reference: Rojas-Perilla et al. (2019).

Using emdi

Currently function `ebp()` includes a logarithmic or Box-Cox transformation

```
# EBP estimation function under a Box-Cox
   transformation
ebp_au <- ebp(fixed = eqIncome ~ gender + eqsize +
                         py010n + py050n + py090n +
                         py100n + py110n + py120n +
                         py130n + hy040n + hy050n +
                         hy070n + hy090n + hy145n,
              pop_data = eusilcP_HH,
              pop_domains = "region",
              smp_data = eusilcS_HH,
              smp_domains = "region",
              pov_line = 0.6*median(eusilcS_HH$eqIncome
                ),transformation = "box.cox",L=50,
               MSE = T,B = 50)
```

Reference: Kreutzmann et al. (2019).

Using emdi - Summary output

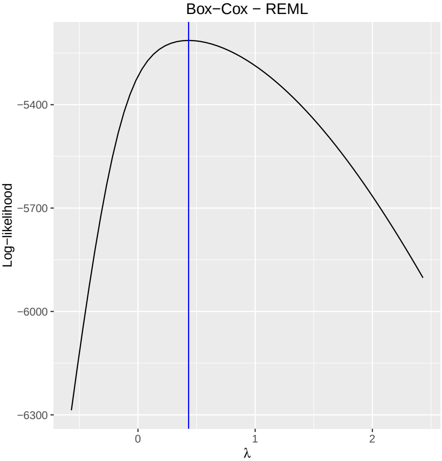
```
# Summary for the EBP method
  > summary(ebp_au)
```

```
Transformation:
 Transformation Method Optimal_lambda Shift_parameter
        box.cox   reml      0.4317972               0

Explanatory measures:
 Marginal_R2 Conditional_R2
   0.4543301      0.4543301

Residual diagnostics:
               Skewness Kurtosis Shapiro_W    Shapiro_p
Error          0.76051    6.3646   0.95643   4.9497e-11
Random_effect 0.58501     2.5533   0.95227   7.1501e-01
```

Reference: Kreutzmann et al. (2019).

Finding $\hat{\lambda}$

Graphical representation of the optimal $\hat{\lambda}$ is made using the function `plot`



Model and Design-based evaluation using Monte-Carlo simulation

- **Model-based evaluation**
  - Uses synthetic data generated under a model
  - Sampling is performed repeatedly from the population generated in each Monte-Carlo round
  - Useful for evaluating performance and sensitivity of new methods under different assumptions
- **Design-based evaluation**
  - Uses frame data (e.g. census data) or synthetic data (not generated under a model) that preserve the survey characteristics
  - Sampling is performed repeatedly by keeping the population fixed
  - Useful for comparing competing methods in more realistic settings

Quality measures - R simulations

Root mean square error:

$$RMSE_i = \sqrt{\frac{1}{R}\sum_{r=1}^{R}\left(\hat{\theta}_{i,r} - \theta_{i,r}\right)^2}$$

Relative bias [%]:

$$RB_i = \frac{1}{R}\sum_{r=1}^{R}\frac{\hat{\theta}_{i,r} - \theta_{i,r}}{\theta_{i,r}} \cdot 100$$

Model-based evaluation

Population data: is generated for $m = 50$ areas with $N = 200$ via

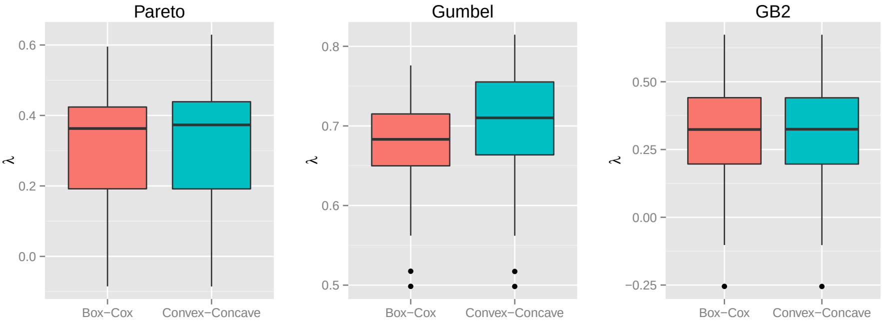
$$y_{ij} = 4500 - 400x_{ij} + u_i + e_{ij}$$

- Covariates $x_{ij} \sim N(\mu_i, 3^2)$ with $\mu_i \sim U(-3, 3)$
- Random effects $u_i \sim N(0, 500^2)$
- Unbalanced design leading to a sample size of $n = 921$ ($min = 8$, $mean = 18.4$, $max = 29$)
- 100 Monte Carlo replicates with L=50 bootstraps

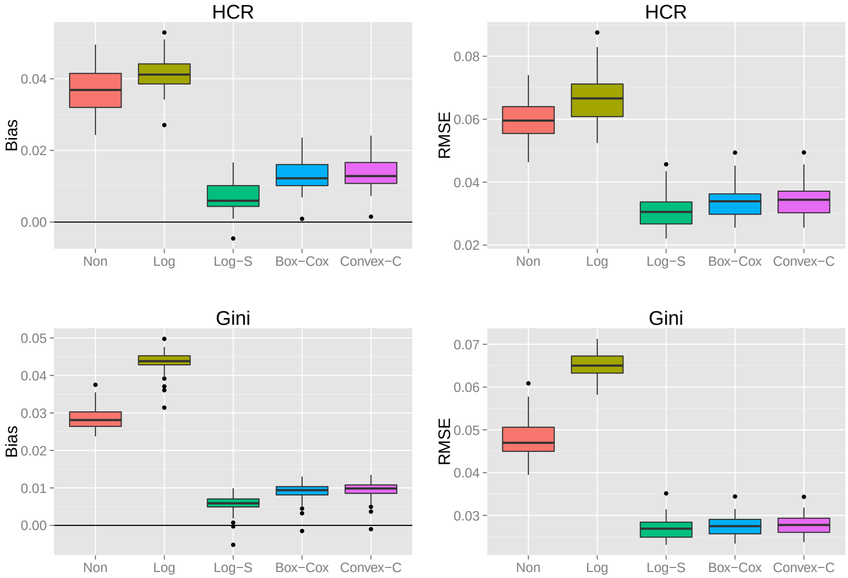
Scenarios: Three different income distribution are investigated:

$$e_{ij} \sim \text{Pareto}(2.5, 100)$$
$$e_{ij} \sim GB2(3, 700, 1, 0.8)$$
$$e_{ij} \sim Gumbel(1, 1000)$$

Estimated transformation parameters



Performance under the Pareto scenario using REML



Design-based evaluation: State of Mexico (EDOMEX)

- **Target geography**: State of Mexico is made up of 125 administrative divisions
- **Survey**: 58 are in-sample and 67 out-of-sample
- **Census**: From the 219514 households, there are 2748 in the sample
- **Sample sizes**:

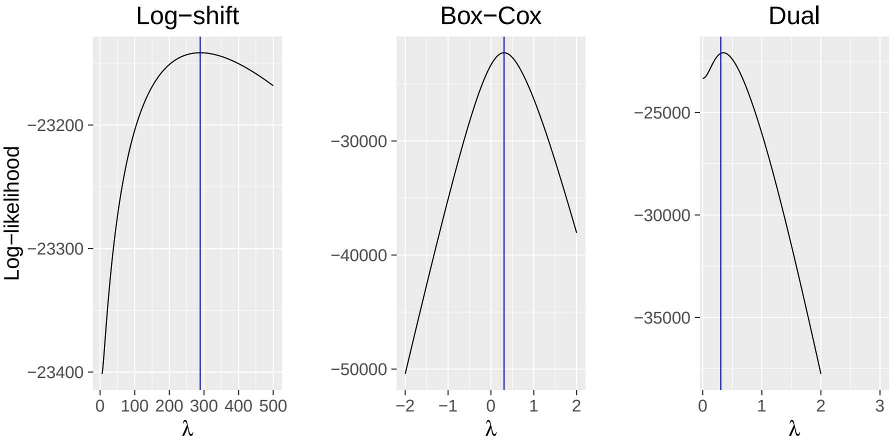
	Min.	Q1.	Median	Mean	Q3	Max.
Survey	3	17	21	47	42	527
Census	650	923	1161	1756	1447	13580

Outcome: Two income variables are available in the survey.
The target variable is available only on the survey. Earned per capita income from work is also available on the Census micro data

Design-based evaluation: Setup

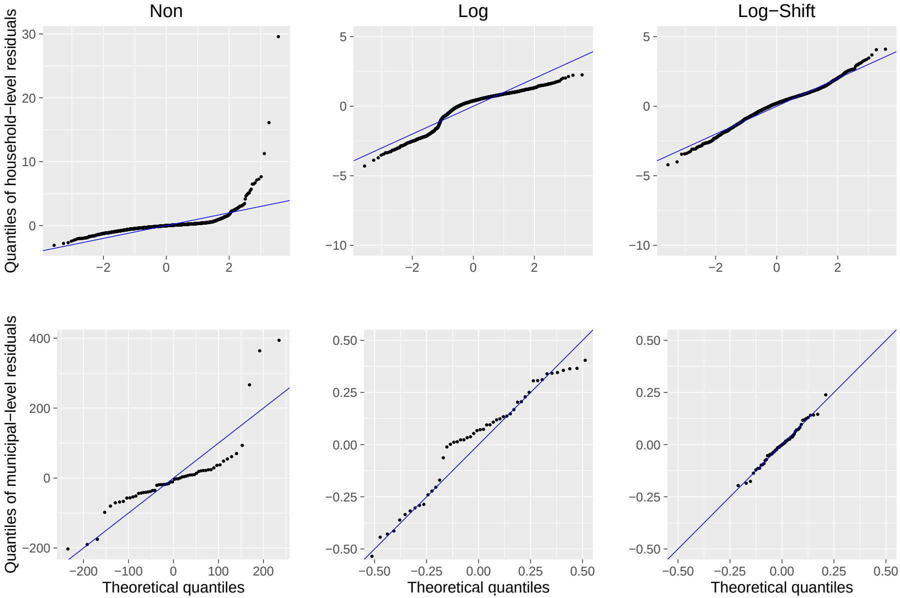
- Design-based simulation with 500 MC-replications repeatedly drawn from EDOMEX Census
- Unbalanced design leading to a sample size of $n = 2195$ ($\min = 8$, $\text{mean} = 17.6$, $\max = 50$)
- Sampling from each municipality

Transformation parameters - Estimation



	Log-shift	Box-Cox	Dual
λ	289.46	0.31	0.35

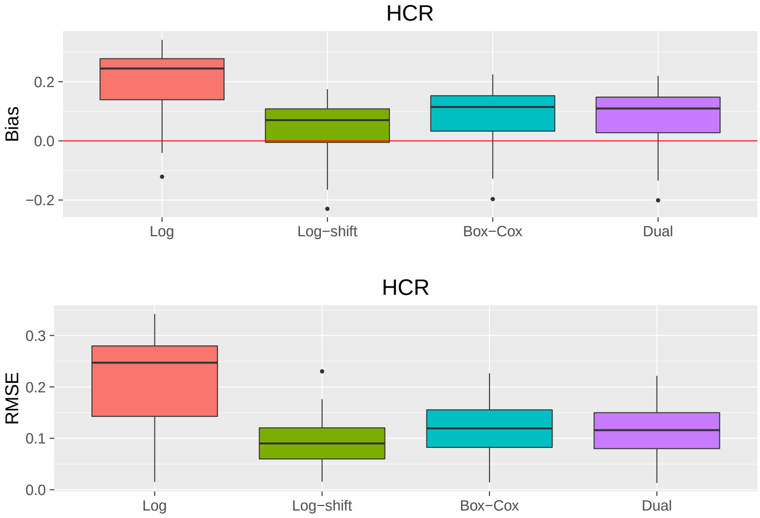
Residual diagnostics



Model diagnostics

Transformation	No	Log	Log-Shift	Box-Cox	Dual
R^2	0.30	0.40	0.52	0.48	0.48
ICC	0.004	0.046	0.032	0.029	0.027

Estimated HCR under alternative transformations



Some further topics

- Methods for discrete outcomes (e.g. binary and count)
- Use of GLMMs
- Outlier robust methods
- Model selection & testing
- Non-parametric models
- Models with spatial structure in the random effects
- Benchmarking methods
- SAE methods with linked data
- SAE methods with interval-censored response data

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